

Notes on SAW Tag Interrogation Techniques

1. Introduction

We consider the problem of interrogating a single SAW RFID tag with a known ID and known range in the presence of multiple interfering tags under the following assumptions:

- The RF propagation environment is well approximated as a simple delay channel with geometric power-decay constant $\alpha \geq 2$.
- The interfering tag IDs are unknown but well approximated as independent, identically distributed random samples from a probability distribution of tag ID waveforms with known second-order properties, and the tag of interest is drawn independently from the same distribution.
- The ranges of the interfering tags are unknown but well approximated as independent, identically distributed realizations of a random variable ρ with a known probability distribution f_ρ , and the tag ranges are independent of the tag ID waveforms.

In particular, we model the tag waveforms as random impulse responses from a wide-sense-stationary, uncorrelated-scattering (WSSUS) fading channel with known bandwidth and scattering function. A brief discussion of the properties of such channels and the notation used to describe them in this document is given in the Appendix.

Under these assumptions, we derive the expression for the output signal-to-noise ratio (SNR) for an arbitrary combination of transmitted interrogation signal and linear receiver filter. Based on this expression, we derive the optimal interrogator configuration (i.e., transmitted signal/receiver filter combination) in the two extreme noise/interference regimes, i.e., noise-limited and interference-limited, under the additional assumption that the coherence bandwidth of the tags is much smaller than the total tag bandwidth. Finally, we evaluate the performance of both optimal interrogators over a broad range of operating scenarios using both numerical simulation based on the assumed model and Monte Carlo simulation based on a small sample of measured tag waveforms. The performance evaluation results not only provide guidelines for proper interrogator design, but also provide some insight on the validity of the assumed signal model.

It should be noted that the assumption that the impulse response of the tag of interest is known precisely implies that the temperature and range of the tag are also known precisely, which is generally not the case in practice. However, analyzing interrogator performance under this simplifying assumption is much more straightforward and still provides a great deal of insight into the nature of the problem.

2. Basic Approach

Let $H_0(f)$ represent the known frequency response function (i.e., the Fourier transform of the impulse response) of the tag of interest, and let $H_k(f)$ for $k=1, 2, \dots, K$,

represent the unknown frequency response functions of the K interfering tags. We assume that all tag frequency response functions satisfy

$$H_k(f) = 0, \quad \forall |f - f_0| > B/2, \quad k = 0, 1, \dots, K,$$

where B is the two-sided bandwidth of the tags and f_0 is an arbitrary center frequency. For the sake of notational simplicity, we will assume for the remainder of this document that we are operating with baseband-equivalent signals, so that $f_0 = 0$. This involves no loss of generality as the results are all valid as stated for arbitrary center frequencies.

If an arbitrary interrogating signal with frequency spectrum $H_T(f)$ and energy \mathcal{E} is transmitted from the transmitting antenna, then the return signal from tag $k = 0, 1, \dots, K$, received at the (co-located) receiving antenna is given by

$$X_k(f) = \sqrt{\frac{\mathcal{E}}{\int_{-B/2}^{B/2} |H_T(f)|^2 df}} \cdot r_k^{-\alpha} H_T(f) H_k(f) e^{-i4\pi f r_k/c},$$

where c is the signal propagation velocity and r_k is the range of the tag from both the transmitting and receiving antennas measured in meters¹. If the receiver implements a filter with frequency response $H_R(f)$ sampled at output time $t = 0$, then the output from the receiver due only to the return signal $X_k(f)$ will be

$$S_k = \mathcal{E} r_k^{-\alpha} \int_{-B/2}^{B/2} H_R(f) \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} H_k(f) e^{-i4\pi f r_k/c} df,$$

where $\mathcal{E}_T = \int_{-B/2}^{B/2} |H_T(f)|^2 df$. Hence, the output from the receiver filter due to the signal of interest is given by

$$S_0 = \mathcal{E} r_0^{-\alpha} \int_{-B/2}^{B/2} H_R(f) \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} H_0(f) e^{-i4\pi f r_0/c} df,$$

and the aggregate interference from all of the interfering tags at the output of the receiver, is given by

¹ Note that the amplitude decay rate for the return signal is $r^{-\alpha}$ rather than $r^{-\alpha/2}$ since this is a two-way or *cooperative radar* channel.

$$\begin{aligned}
I &= \sum_{k=1}^K S_k \\
&= \mathcal{E} \sum_{k=1}^K r_k^{-\alpha} \int_{-B/2}^{B/2} H_R(f) \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} H_k(f) e^{-i4\pi f r_k/c} df \\
&= \mathcal{E} \int_{-B/2}^{B/2} H_R(f) \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} \left[\sum_{k=1}^K r_k^{-\alpha} e^{-i4\pi f r_k/c} H_k(f) \right] df.
\end{aligned}$$

Note that the aggregate interference represents a random variable with mean zero and variance

$$\begin{aligned}
\sigma_I^2 &= \int_{-B/2}^{B/2} \int_{-B/2}^{B/2} H_R(f) \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} \bar{H}_R(\lambda) \frac{\bar{H}_T(\lambda)}{\sqrt{\mathcal{E}_T}} \\
&\quad \cdot \left[\sum_{k=1}^K \sum_{j=1}^K \mathbb{E} \left\{ r_k^{-\alpha} r_j^{-\alpha} e^{-i4\pi f r_k/c} e^{i4\pi \lambda r_j/c} \right\} \mathbb{E} \left\{ H_k(f) \bar{H}_j(\lambda) \right\} \right] df d\lambda \\
&= \mathcal{E} \int_{-B/2}^{B/2} \int_{-B/2}^{B/2} H_R(f) \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} \bar{H}_R(\lambda) \frac{\bar{H}_T(\lambda)}{\sqrt{\mathcal{E}_T}} \\
&\quad \cdot \varphi(f - \lambda) \left[\sum_{k=1}^K \mathbb{E} \left\{ r_k^{-2\alpha} e^{-i4\pi(f-\lambda)r_k/c} \right\} \right] df d\lambda \\
&= K \mathcal{E} \int_{-B/2}^{B/2} \int_{-B/2}^{B/2} H_R(f) \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} \bar{H}_R(\lambda) \frac{\bar{H}_T(\lambda)}{\sqrt{\mathcal{E}_T}} \varphi(f - \lambda) \rho(f - \lambda) df d\lambda,
\end{aligned}$$

where $\varphi(f - \lambda)$ is the *spaced-frequency correlation function* (see Appendix) for the tags, and

$$\rho(f - \lambda) = \mathbb{E} \left\{ \rho^{-2\alpha} e^{-i2\pi(f-\lambda)\rho/c} \right\}$$

represents a similar spaced-frequency correlation function for the multiplicative power decay and phase shift of the interfering tags.

Finally, the output from the receiver filter due to a complex-valued additive white Gaussian noise (AWGN) process with power spectral density N_0 and associated orthogonal increments frequency-domain process $Z(f)$ is given by

$$N = \int_{-B/2}^{B/2} H_R(f) Z(df),$$

which is a Gaussian random variable with mean zero and variance

$$\sigma_N^2 = N_0 \int_{-B/2}^{B/2} |H_R(f)|^2 df.$$

Hence, if we let $Y = S_0 + I + N$ represent the random output from the receiver filter in the presence of the signal of interest, additive interference, and AWGN, then it follows that the output SNR at the receiver is given by

$$\begin{aligned}
 SNR_{I+N} &= \frac{E\{Y\}}{\text{Var}\{Y\}} = \frac{|S_0|^2}{\sigma_N^2 + \sigma_I^2} \\
 &= \frac{\mathcal{E}r_0^{-2\alpha} \left| \int_{-B/2}^{B/2} H_R(f) \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} H_0(f) e^{-i4\pi f r_0/c} df \right|^2}{K\mathcal{E} \int_{-B/2}^{B/2} \int_{-B/2}^{B/2} \left[\frac{H_R(f) \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} \bar{H}_R(\lambda) \frac{\bar{H}_T(\lambda)}{\sqrt{\mathcal{E}_T}}}{\cdot \phi(f-\lambda) \rho(f-\lambda)} \right] df d\lambda + N_0 \int_{-B/2}^{B/2} |H_R(f)|^2 df}. \quad (1)
 \end{aligned}$$

Assuming that the aggregate interference is well approximated as Gaussian, the detection performance of the interrogator is determined by this quantity. Such an assumption is probably only justified for a large number of interfering tags, but maximizing the output SNR remains a worthwhile optimization criterion in any case.

As a rule, Expression (1) can be simplified quite a bit by making the following additional assumptions, which should be valid approximately for tags with good anti-collision properties and appropriate choices of $H_T(f)$ and $H_R(f)$:

1. $\phi(v) = 0 \ \forall |v| > \Delta/2$, where $\Delta \ll B$. Recall that Δ represents the *coherence bandwidth* for the set of tag waveforms,
2. $H_R(f) \approx H_R(\lambda)$, $H_T(f) \approx H_T(\lambda) \ \forall |f - \lambda| < \Delta$,
3. $H_k(f) \approx H_k(\lambda) \ \forall |f - \lambda| < \Delta, k = 0, 1, \dots, K$.

Under these assumptions, the expression for the variance becomes

$$\begin{aligned}
 \sigma_I^2 &\approx K\mathcal{E} \int_{-B/2}^{B/2} \left| H_R(\lambda) \frac{H_T(\lambda)}{\sqrt{\mathcal{E}_T}} \right|^2 \int_{\lambda-\Delta/2}^{\lambda+\Delta/2} \phi(f-\lambda) \rho(f-\lambda) df d\lambda \\
 &= K\mathcal{E} \int_{-B/2}^{B/2} \left| H_R(\lambda) \frac{H_T(\lambda)}{\sqrt{\mathcal{E}_T}} \right|^2 d\lambda \cdot \int_{-\Delta/2}^{\Delta/2} \phi(v) \rho(v) dv,
 \end{aligned}$$

which is a considerable simplification, and the expression for the output SNR becomes

$$\begin{aligned}
\text{SNR}_{I+N} &\approx \frac{\mathcal{E} r_0^{-2\alpha} \left| \int_{-B/2}^{B/2} H_R(f) \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} H_0(f) e^{-i4\pi f r_0/c} df \right|^2}{K \mathcal{E} \int_{-B/2}^{B/2} \left| H_R(f) \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} \right|^2 df \cdot \int_{-\Delta/2}^{\Delta/2} \varphi(v) \rho(v) dv + N_0 \int_{-B/2}^{B/2} |H_R(f)|^2 df} \\
&= \frac{\mathcal{E} r_0^{-2\alpha} \left| \int_{-B/2}^{B/2} H_R(f) \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} H_0(f) e^{-i4\pi f r_0/c} df \right|^2}{K \mathcal{E} N_I \int_{-B/2}^{B/2} \left| H_R(f) \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} \right|^2 df + N_0 \int_{-B/2}^{B/2} |H_R(f)|^2 df},
\end{aligned}$$

where $N_I = \int_{-\Delta/2}^{\Delta/2} \varphi(v) \rho(v) dv$. Note that as $K \mathcal{E} N_I / N_0 \rightarrow \infty$, the interference term in the denominator dominates, and this expression simplifies to

$$\text{SNR}_I = \frac{r_0^{-2\alpha}}{K N_I} \cdot \frac{\left| \int_{-B/2}^{B/2} H_R(f) H_T(f) H_0(f) e^{-i4\pi f r_0/c} df \right|^2}{\int_{-B/2}^{B/2} |H_R(f) H_T(f)|^2 df}. \quad (2)$$

On the other hand, as $K \mathcal{E} N_I / N_0 \rightarrow 0$, the AWGN term dominates, and the expression simplifies to

$$\text{SNR}_N = r_0^{-2\alpha} \frac{\mathcal{E}}{N_0} \cdot \frac{\left| \int_{-B/2}^{B/2} H_R(f) \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} H_0(f) e^{-i4\pi f r_0/c} df \right|^2}{\int_{-B/2}^{B/2} |H_R(f)|^2 df}. \quad (3)$$

3. Optimal Interrogation

Note that in the general case, the choice of the transmitted signal $H_T(f)$ and the receiver filter $H_R(f)$ that maximize the output SNR given by Expression (1) must be determined numerically by solving a fairly complex variational problem. Hence, in general, the optimal interrogator design and the corresponding detection performance is rather difficult to determine. However, for practical purposes, the problem can often be reduced to either the interference-limited case or the noise-limited case². In each of the two limiting cases, the optimal interrogator design and corresponding optimal output SNR are easy to determine. We consider each case separately below.

² The problem reduces to the interference-limited case as the product of the number of interfering tags and the power of the transmitted signal goes to infinity. As the same product goes to zero, the problem reduces to the noise-limited case.

3.1. Noise-Limited Case

For this case, the output SNR is given by Expression (3) and satisfies

$$\begin{aligned} SNR_N &= r_0^{-2\alpha} \frac{\mathcal{E}}{N_0} \cdot \frac{\left| \int_{-B/2}^{B/2} H_R(f) \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} H_0(f) e^{-i4\pi f r_0/c} df \right|^2}{\int_{-B/2}^{B/2} |H_R(f)|^2 df} \\ &\leq r_0^{-2\alpha} \frac{\mathcal{E}}{N_0} \cdot \int_{-B/2}^{B/2} \left| \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} H_0(f) \right|^2 df. \end{aligned}$$

It follows from the Cauchy-Schwarz (C-S) inequality that equality is achieved in this expression if and only if

$$H_R(f) = \beta \frac{\bar{H}_T(f)}{\sqrt{\mathcal{E}_T}} \bar{H}_0(f) e^{i4\pi f r_0/c},$$

for an arbitrary complex constant $\beta \neq 0$. Furthermore, in this case, we have

$$SNR_N = r_0^{-2\alpha} \frac{\mathcal{E}}{N_0} \cdot \int_{-B/2}^{B/2} \left| \frac{H_T(f)}{\sqrt{\mathcal{E}_T}} H_0(f) \right|^2 df \leq r_0^{-2\alpha} \frac{\mathcal{E}}{N_0} \cdot \int_{-B/2}^{B/2} |H_0(f)|^2 df,$$

where equality is achieved if and only if

$$H_T(f) = \gamma \bar{H}_0(f),$$

for an arbitrary constant $\gamma \neq 0$. Hence, without loss of generality, the optimal interrogating signal satisfies

$$H_T(f) = \bar{H}_0(f),$$

the optimal receiver filter satisfies

$$H_R(f) = \beta \frac{|H_0(f)|^2}{\sqrt{\mathcal{E}_0}} e^{i4\pi f r_0/c},$$

where $\mathcal{E}_0 = \int_{-B/2}^{B/2} |H_0(f)|^2 df$ and $\beta \neq 0$ is arbitrary, and the optimal output SNR is given by

$$SNR_N^* = r_0^{-2\alpha} \frac{\mathcal{E}}{N_0} \cdot \int_{-B/2}^{B/2} |H_0(f)|^2 df.$$

Finally, averaging over the constellation of random tag waveforms gives

$$\overline{SNR_N^*} = r_0^{-2\alpha} \frac{\mathcal{E}}{N_0} \cdot \mathbb{E} \left\{ \int_{-B/2}^{B/2} |H_0(f)|^2 df \right\} = r_0^{-2\alpha} B \varphi(0) \frac{\mathcal{E}}{N_0}.$$

Notice that in this case, the optimal transmitted signal is just the time-reverse of the impulse response of the interrogating signal for the tag of interest. This is not surprising given the “well-known” optimality properties of time-reversal processing [1, 2].

3.2. Interference-Limited Case

For this case, the output SNR is given by Expression (2) and satisfies

$$SNR_I = \frac{r_0^{-2\alpha}}{KN_I} \cdot \frac{\left| \int_{-B/2}^{B/2} H_R(f) H_T(f) H_0(f) e^{-i4\pi f r_0/c} df \right|^2}{\int_{-B/2}^{B/2} |H_R(f) H_T(f)|^2 df} \leq \frac{r_0^{-2\alpha}}{KN_I} \int_{-B/2}^{B/2} |H_0(f)|^2 df.$$

Again, it follows from the C-S inequality that equality is achieved in this expression if and only if

$$H_R(f) H_T(f) = \beta \bar{H}_0(f) e^{i4\pi f r_0/c},$$

for an arbitrary complex constant $\beta \neq 0$. Hence, one possible choice for the optimal interrogating signal satisfies

$$H_T(f) = \begin{cases} 1, & |f| \leq B/2, \\ 0, & |f| > B/2, \end{cases}$$

the corresponding optimal receiver filter satisfies

$$H_R(f) = \beta \bar{H}_0(f) e^{i4\pi f r_0/c},$$

where $\beta \neq 0$ is arbitrary, and the optimal output SNR is given by

$$SNR_I^* = \frac{r_0^{-2\alpha}}{KN_I} \int_{-B/2}^{B/2} |H_0(f)|^2 df.$$

Finally, averaging over the constellation of random tag waveforms gives

$$\overline{SNR_I}^* = \frac{r_0^{-2\alpha}}{KN_I} \mathbb{E} \left\{ \int_{-B/2}^{B/2} |H_0(f)|^2 df \right\} = \frac{r_0^{-2\alpha}}{KN_I} B \varphi(0).$$

Notice that in this case, the optimal transmitted signal is just a band-limited impulse function, which can be approximated using either a stepped CW signal or an FM chirp.

4. Performance Evaluation

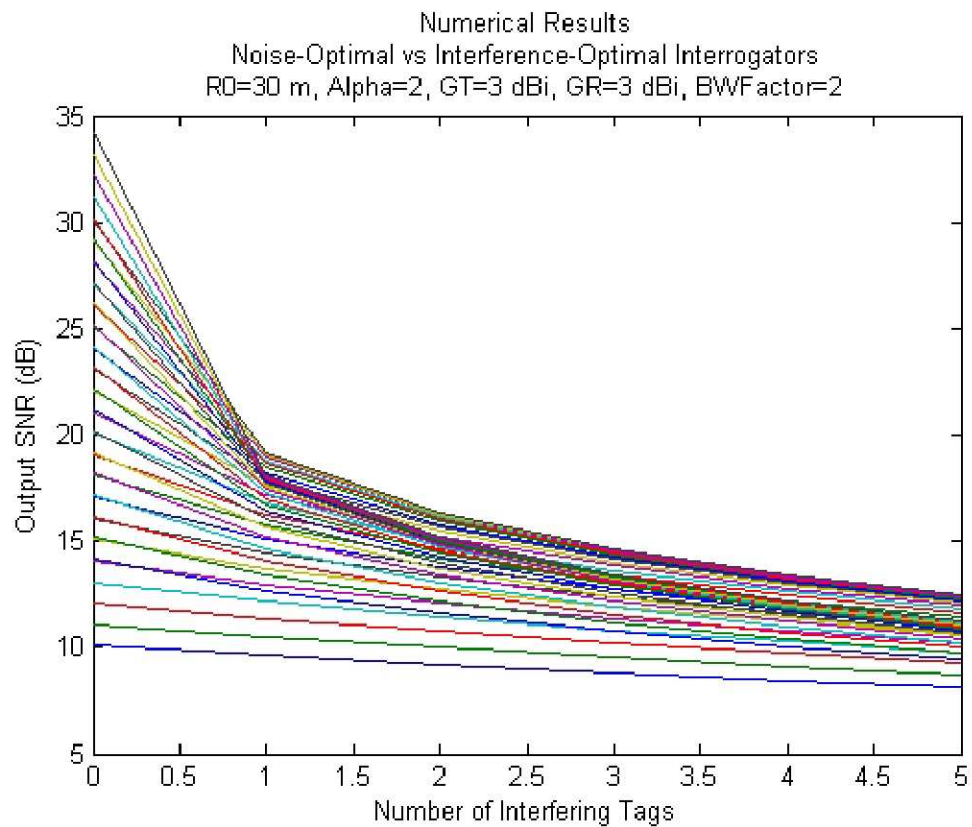
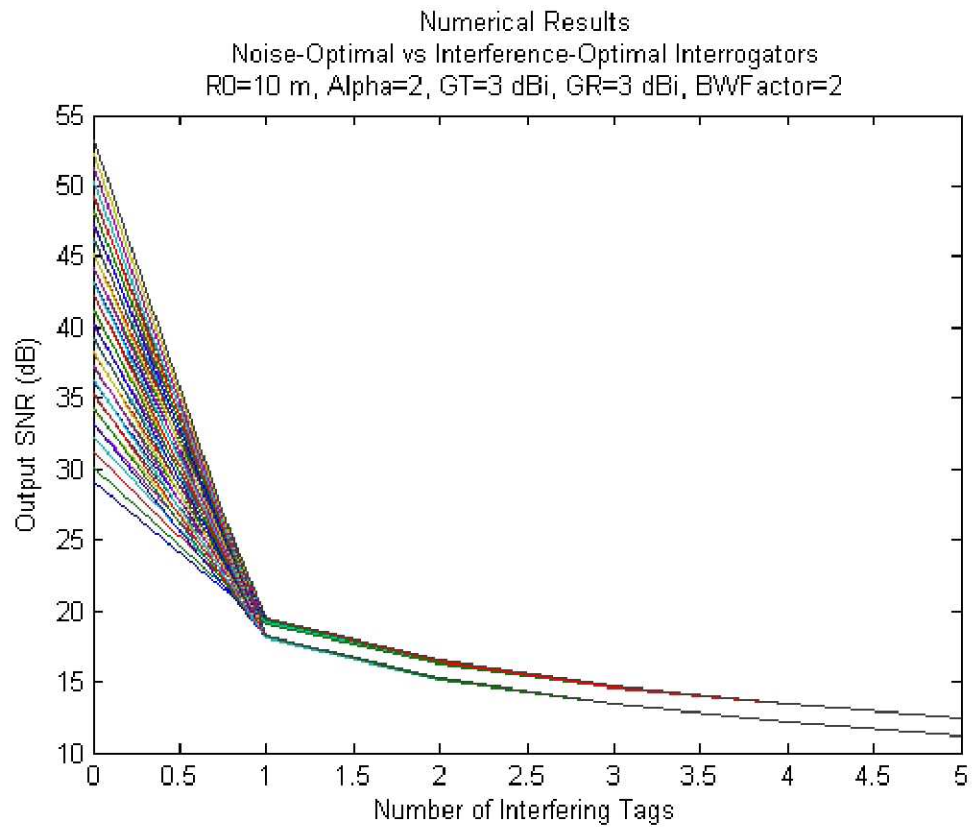
To compare the performance of the two interrogator configurations derived in Sections 3.1 and 3.2 above (which would be optimal in the noise-limited case and the interference-limited case, respectively, if our assumptions were satisfied), we computed the value of the output SNR for each configuration both numerically and using Monte Carlo simulation over a wide range of operational scenarios. All performance results were derived from a collection of 23 measured tag frequency response vectors that were generated by sampling the actual frequency response of 23 SAW RFID tags chosen randomly from a homogenous population of tags provided by a single manufacturer.

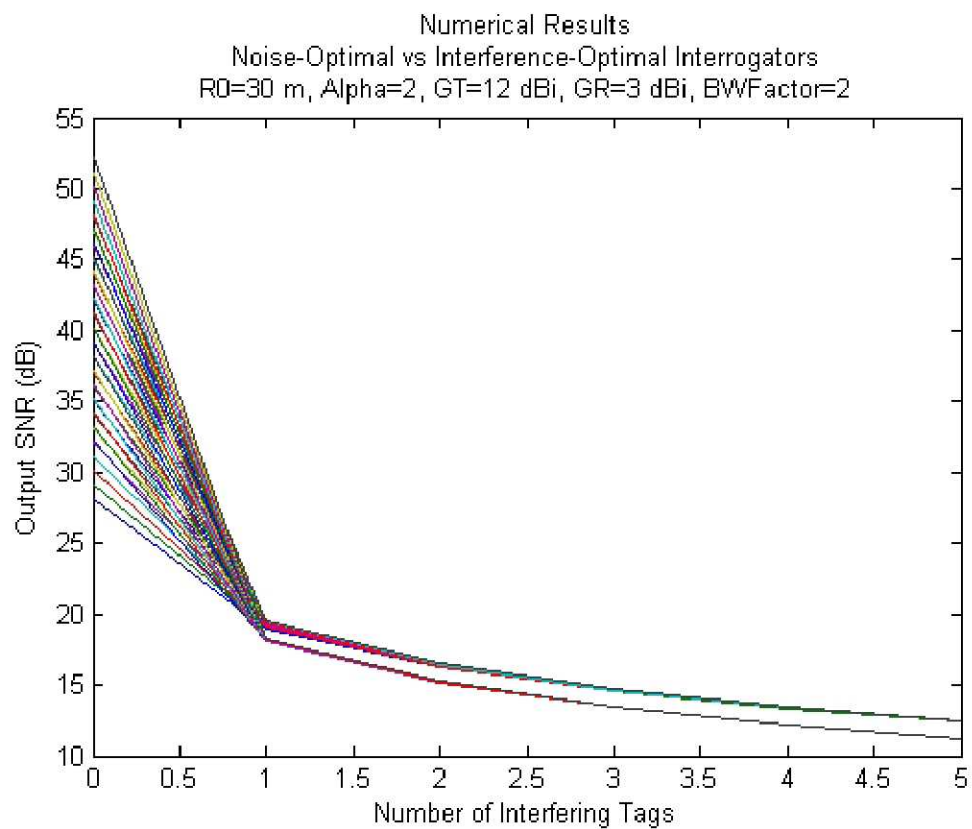
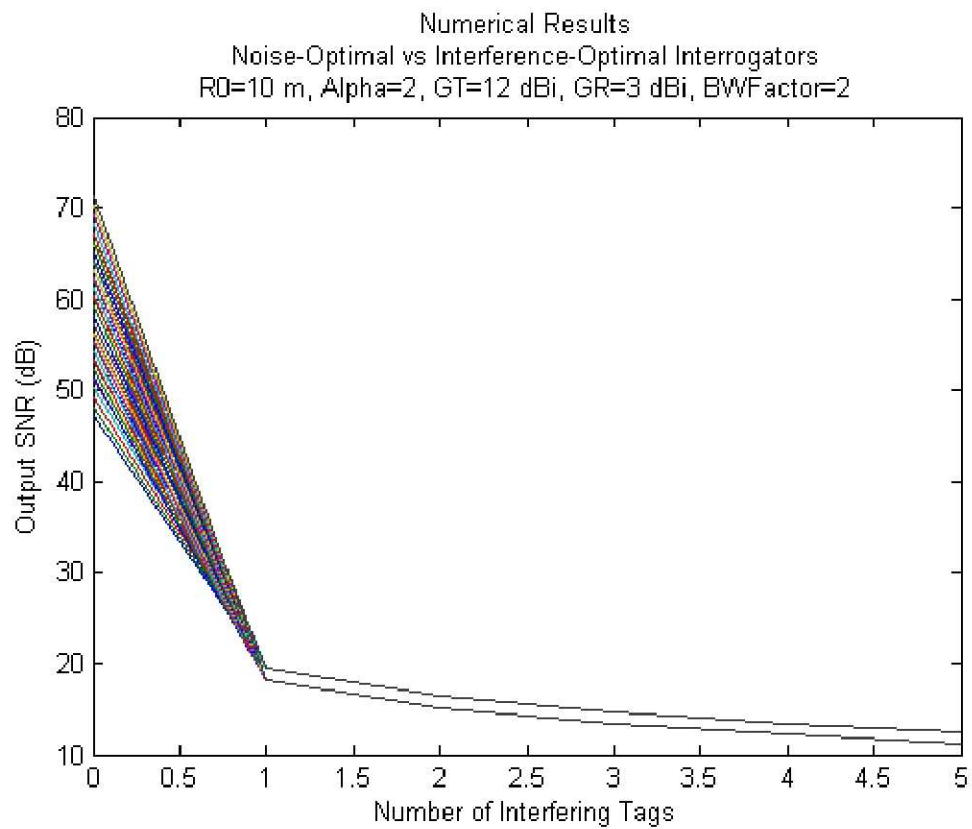
Numerical performance results were derived by evaluating Expression (1) numerically using a sampled version of $\phi(f - \lambda)$ estimated directly from the population of tag frequency response vectors. Simulated performance was derived by generating random realizations of the random variable $Y = S_0 + I + N$ that underlies Expression (1), directly estimating the quantities $E\{Y\}$ and $\text{Var}\{Y\}$, and computing an estimate of output SNR as $E\{Y\}/\text{Var}\{Y\}$.

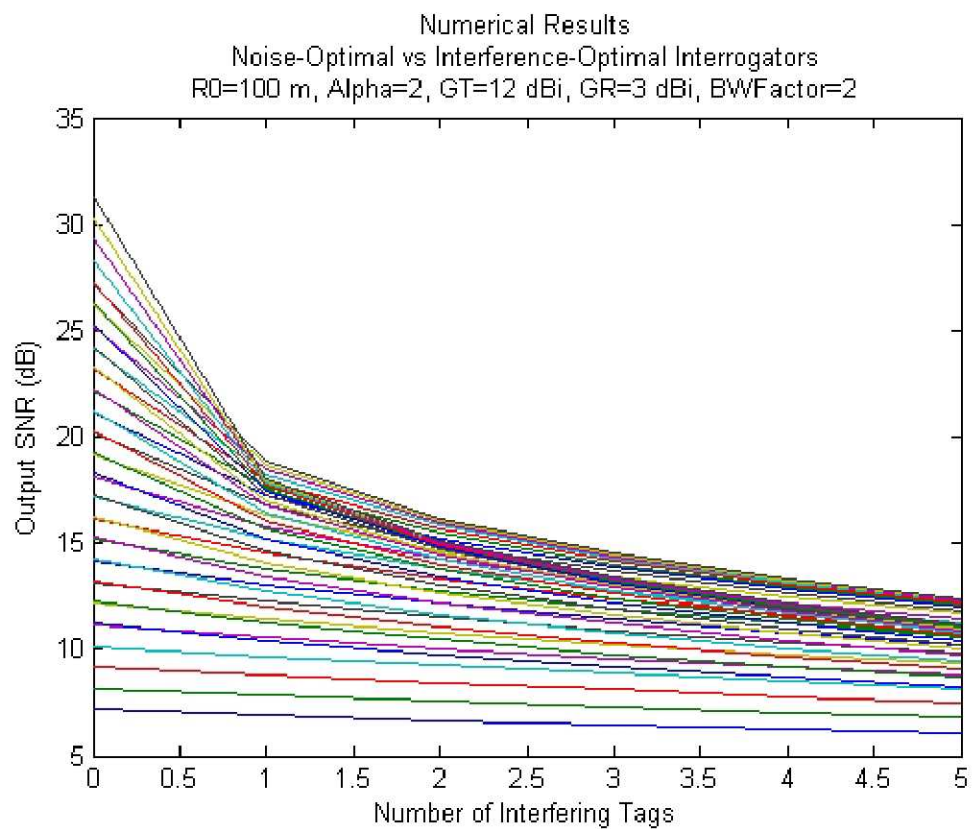
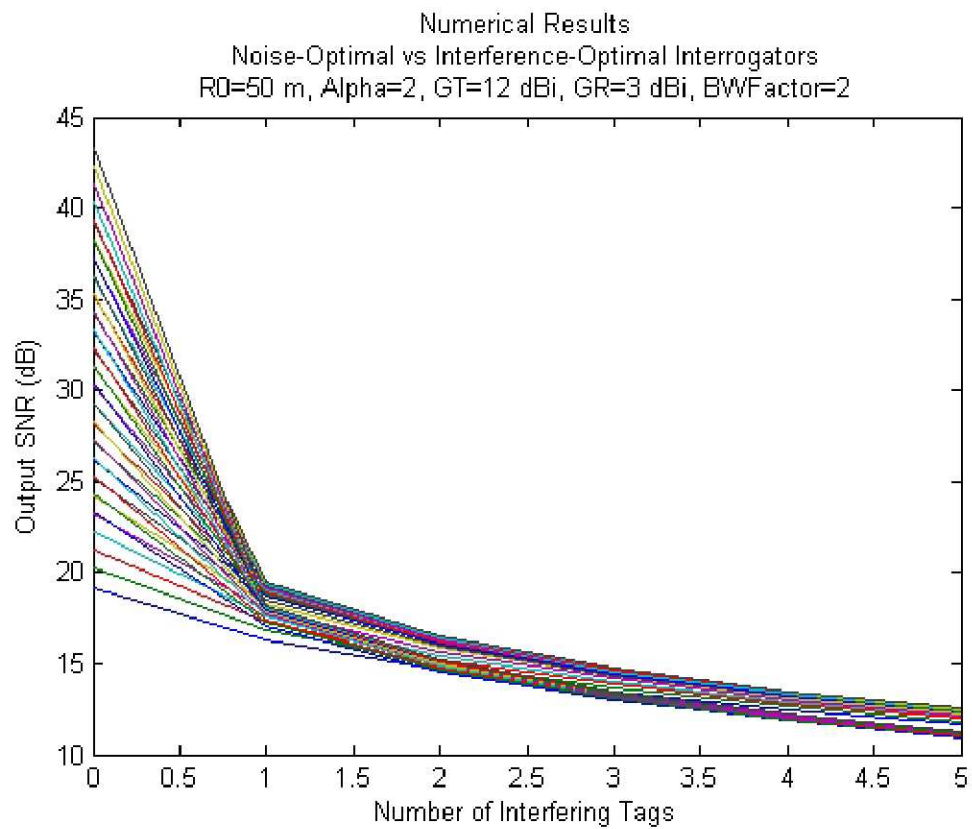
For the performance results presented in the following two sections, all tags (i.e., the tag of interest as well as all interfering tags) were assumed to be at the same range, represented by the variable R_0 , propagation was assumed to be in free space (i.e., $\alpha = 2$), the tag processing bandwidth was fixed at 80 MHz (indicated in the figure titles by the value of $BW_{Factor} = 2$), and the tag antenna temperature was assumed to be 300° Kelvin. The assumed gain of the transmitting antenna is represented by the variable G_T , the gain of the receiving antenna is represented by the variable G_R , and the output SNR curves in the figures correspond to 21 different transmitted power levels uniformly spaced at 1 dBm increments from 10 dBm to 30 dBm with a fixed receiver processing bandwidth (i.e., IF bandwidth) of 15 kHz. Finally, all tag frequency response waveforms were scaled to correspond to an insertion loss of approximately 30 dB in total tag input vs. output energy. It follows that for all figures, the output SNR value for the *interference-optimal* (i.e., band-limited impulse) interrogator with no interfering tags corresponds exactly to the value of SNR at the input to the receiver (i.e., E_b/N_0) computed using a traditional link-budget calculation.

4.1. Numerical Performance Results.

For all figures in this section, the 21 curves corresponding to the *noise-optimal* (i.e., time-reversal) interrogator are approximately 4 dB above the corresponding interference-optimal interrogator curves when no interfering tags are present, but drop below the interference-optimal curves in many cases with five interfering tags. If the curves were extended to 100 interfering tags (excluded due to space limitations), both the noise-optimal and the interference-optimal curves would be essentially independent of transmitted power near the 100-tag extreme (i.e., only one curve for each instead of 21) and the noise-only curves would all fall below the corresponding interference-optimal curves at some point in the 0-100 interfering-tag range. This confirms that the noise-optimal interrogator consistently outperforms the interference-optimal interrogator in the noise-limited regime and vice-versa in the interference-limited regime. Interestingly, in many cases, the interference-limited regime is reached with only one interfering tag present.

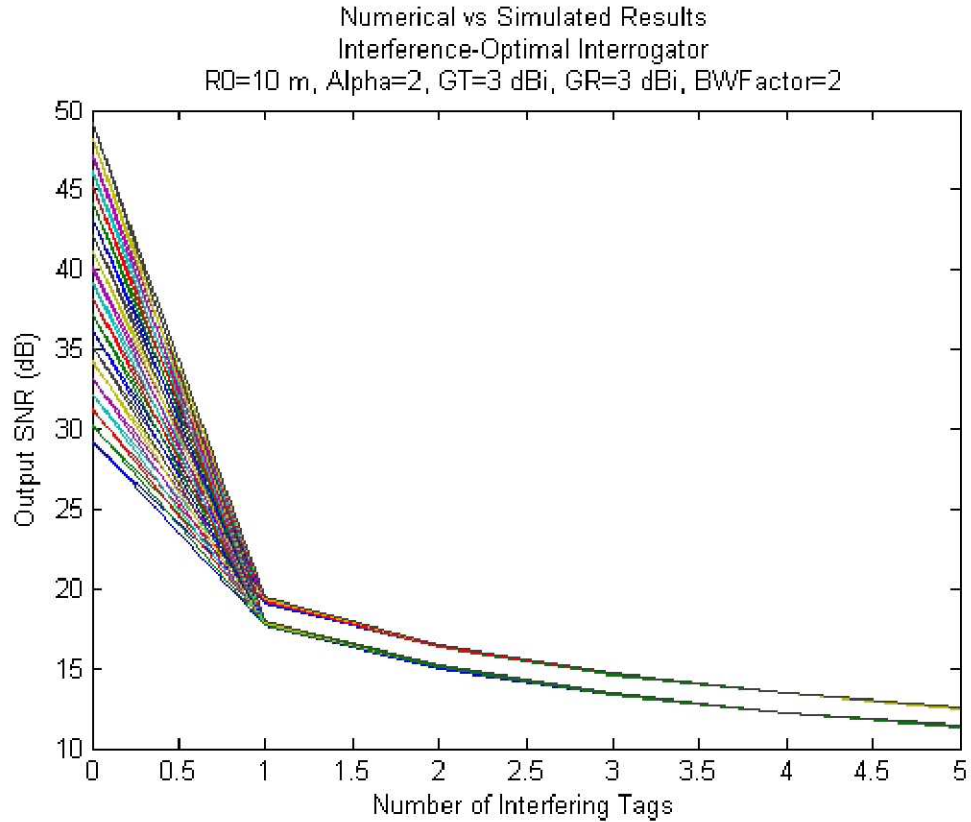


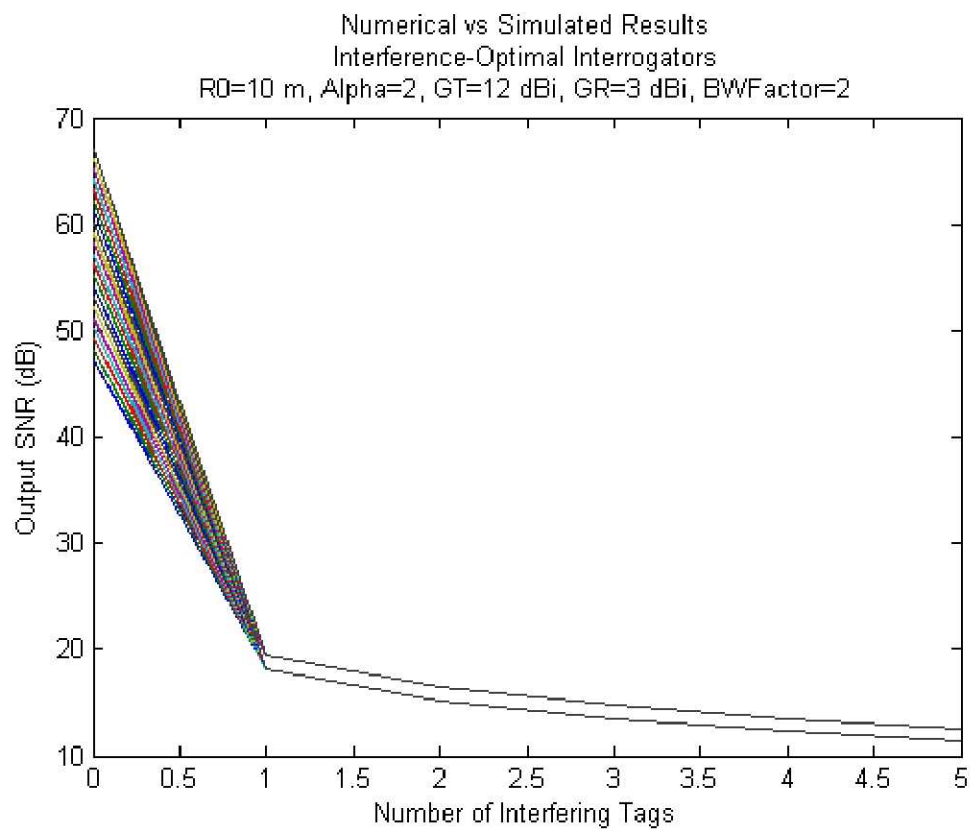
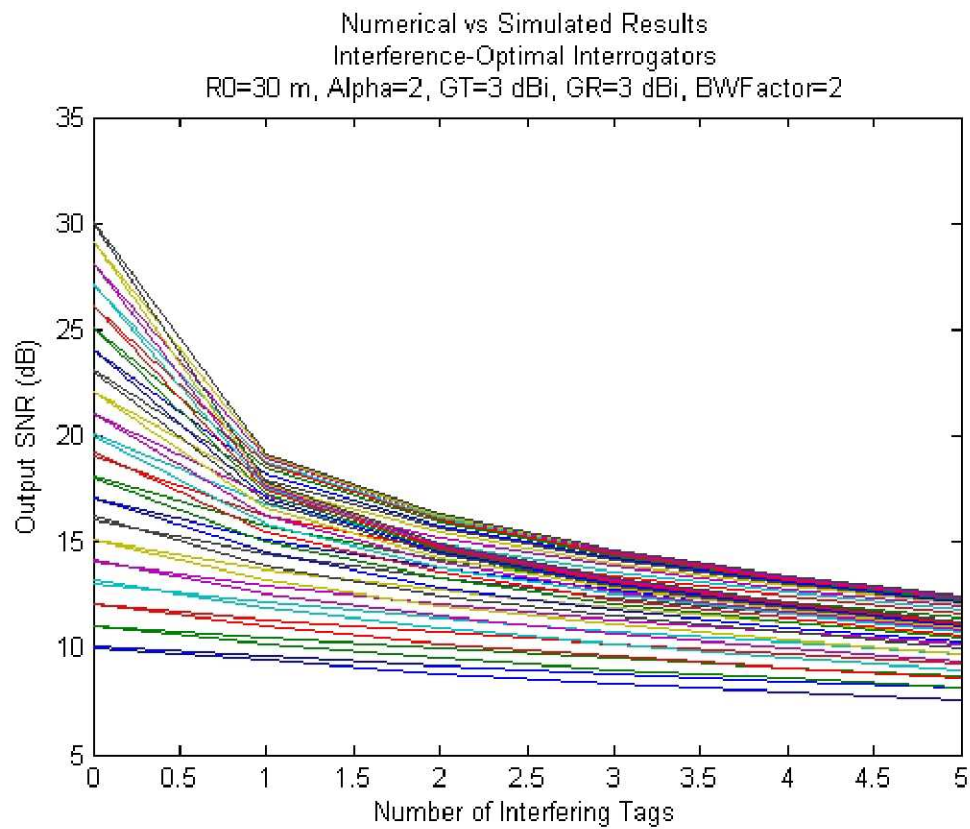


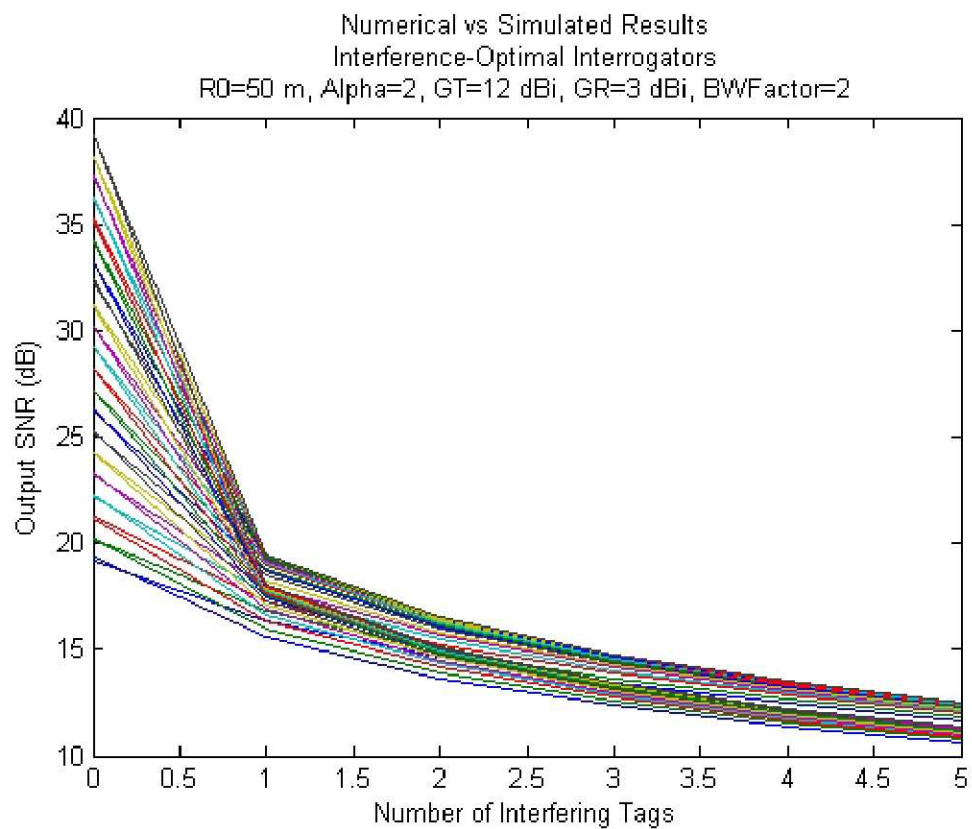
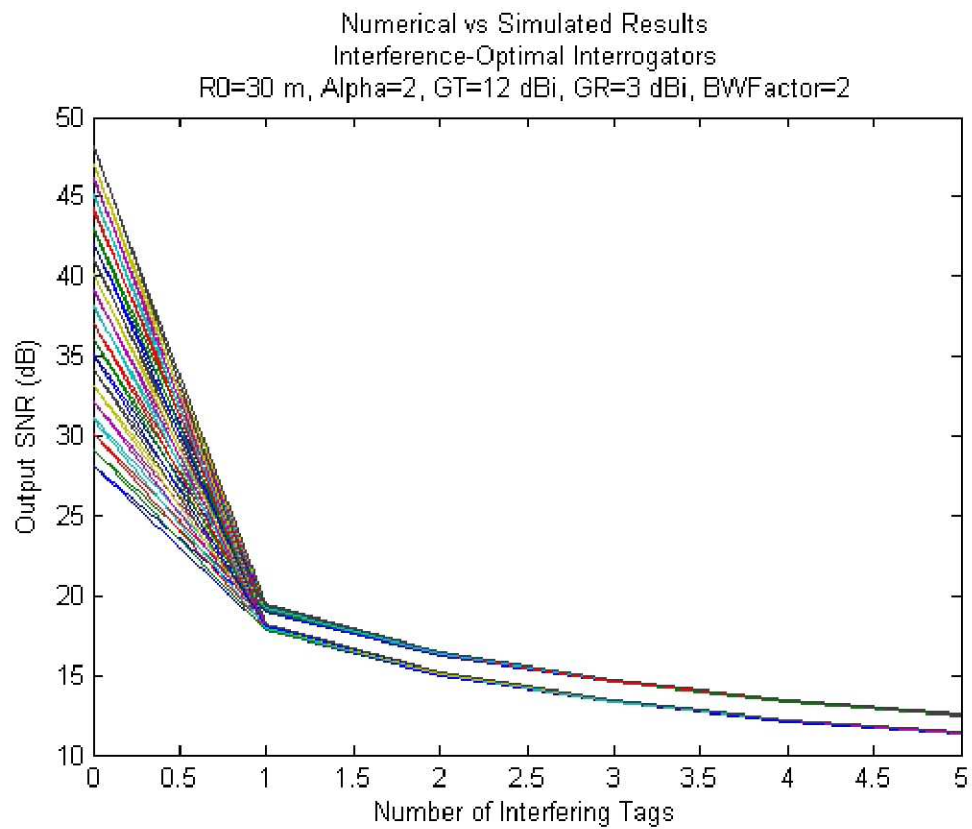


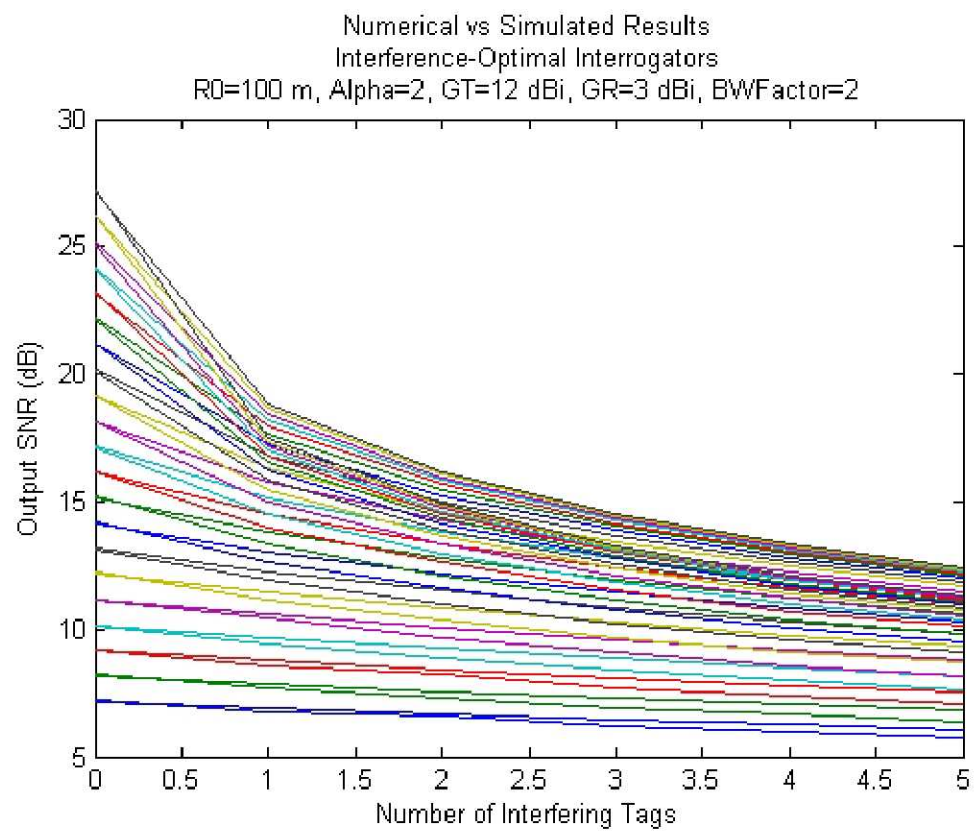
4.2. Numerical vs. Simulated Performance Results.

In this section, we compare the numerical performance results with the simulated performance results as an indication of how well the assumed statistical tag-response model captures the behavior of the small number of measured tag responses in our sample. In all cases, the modeled behavior was nearly identical to the simulated behavior in the absence of interfering tags (as one would definitely expect), but the modeled output SNR always slightly exceeded the simulated SNR when interfering tags were present. Nevertheless, considering the very small size of our measured tag sample, the results were in remarkably good agreement throughout the range of 0-5 interfering tags.









Appendix. Background on Wide-Sense-Stationary Uncorrelated-Scattering Fading Channels

In this appendix, we review the basic concepts and notation for the wide-sense-stationary, uncorrelated-scattering (WSSUS) statistical model of multipath fading channels that is utilized in the previous sections of this document. For a much more thorough discussion, see, for example, [3].

We assume that the channel under consideration has a time-varying impulse response given by $h(\tau; t)$, where t represents the time at which the output is measured and τ represents the delay of the measurement. That is, $h(\tau; t)$ represents the output of the channel at time t due to an impulse transmitted at time $t - \tau$. It is further assumed that $h(\tau; t)$ is a zero-mean, complex random process that is causal, stationary in t , and satisfies the uncorrelated scattering assumption; *i.e.*,

$$\mathbb{E}\{h(\tau; t)\} = 0 \quad \forall t, \tau,$$

and

$$\mathbb{E}\{h(\tau_1; t_1)\bar{h}(\tau_2; t_2)\} = \begin{cases} \hat{\phi}(\tau_1; t_1 - t_2)\delta(\tau_1 - \tau_2), & \tau_1 \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where an overbar indicates complex conjugation, and $\delta(\tau)$ represents the Dirac delta function. Note that if we want to avoid δ functions, we can express everything in the delay frequency domain. That is, if we define the Fourier transform of $h(\tau; t)$ with respect to τ as

$$H(f; t) = \int_{-\infty}^{\infty} h(\tau; t) e^{-i2\pi f \tau} d\tau,$$

then we get

$$\begin{aligned} \mathbb{E}\{H(f_1; t_1)\bar{H}(f_2; t_2)\} &= \int_0^{\infty} \int_0^{\infty} \mathbb{E}\{h(\tau_1; t_1)\bar{h}(\tau_2; t_2)\} e^{-i2\pi f_1 \tau_1} e^{i2\pi f_2 \tau_2} d\tau_1 d\tau_2 \\ &= \int_0^{\infty} \hat{\phi}(\tau_1; t_1 - t_2) e^{-i2\pi \tau_1 (f_1 - f_2)} d\tau_1 \\ &= \phi(f_1 - f_2; t_1 - t_2), \end{aligned}$$

where ϕ represents the so-called *spaced-time, spaced-frequency correlation function*. Note that ϕ is only a function of the difference in both time and frequency; that is, the random process $H(f; t)$ is stationary in both variables. If the explicit output time is not of interest, the t is frequently dropped in favor of the short-hand notation $H(f)$. Similarly, when $t_1 - t_2 = 0$, the notation $\phi(f_1 - f_2; 0)$ is usually replaced by $\phi(f_1 - f_2)$, which is then referred to simply as the *spaced frequency correlation function*.

Note also that φ is often described in terms of its 2-D Fourier transform, which is given by

$$\mathcal{S}(\tau; \lambda) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(f; t) e^{i2\pi\tau f} e^{-i2\pi\lambda t} df dt$$

The function $\mathcal{S}(\tau; \lambda)$ is generally referred to as the *scattering function* of the channel and represents the power spectral density of the two-dimensional stationary random process $H(f; t)$. Finally, it should be noted that we make the assumption of zero mean primarily for simplicity in exposition, but it is realistic in many cases, particularly if the channel has no line-of-sight component. In any case, extending the results to account for nonzero mean is straightforward.

Now, if we transmit a signal $s(t)$ over such a channel, then, ignoring additive noise, the output is a zero-mean, complex random process of the form

$$r(t) = \int_{-\infty}^{\infty} h(\tau; t) s(t - \tau) d\tau,$$

The autocorrelation function of this process is given by

$$\begin{aligned} E\{r(t_1)\bar{r}(t_2)\} &= E\left\{\left[\int_{-\infty}^{\infty} h(\tau_1; t_1) s(t_1 - \tau_1) d\tau_1\right] \left[\int_{-\infty}^{\infty} \bar{h}(\tau_2; t_2) \bar{s}(t_2 - \tau_2) d\tau_2\right]\right\} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E\{h(\tau_1; t_1) \bar{h}(\tau_2; t_2)\} s(t_1 - \tau_1) \bar{s}(t_2 - \tau_2) d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty} \hat{\varphi}(\tau_1; t_1 - t_2) s(t_1 - \tau_1) \bar{s}(t_2 - \tau_1) d\tau_1, \end{aligned}$$

Note that this autocorrelation function implies that the output is generally a nonstationary process unless the signal has some special properties. For example, if $s(t)$ is itself a zero mean, stationary random process that is independent of the channel process, then

$$\begin{aligned} E\{r(t_1)\bar{r}(t_2)\} &= \int_{-\infty}^{\infty} \hat{\varphi}(\tau_1; t_1 - t_2) E\{s(t_1 - \tau_1) \bar{s}(t_2 - \tau_1)\} d\tau_1 \\ &= \varphi_s(t_1 - t_2) \cdot \left[\int_{-\infty}^{\infty} \hat{\varphi}(\tau_1; t_1 - t_2) d\tau_1\right], \end{aligned}$$

where $\varphi_s(t_1 - t_2)$ represents the autocorrelation function of the signal.

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